CHAPTER 5

EIGENVALUES: The Eigenvalues of an $n \times n$ matrix A are the solutions of the Characteristic Equ. $\boxed{\det(\lambda I - A) = 0}$

NOTE 1 The Characteristic Equ. of an $n \times n$ matrix is of degree 'n' and so an $n \times n$ matrix has at most 'n' distinct eigenvalues.

NOTE 2 The Sum of eigenvalues of a square matrix is equal to its Trace.

NOTE 3 The Product of eigenvalues of a square matrix is equal to its Determinant.

THEOREM If 'A' is $n \times n$ triangular matrix (upper triangular, lower triangular or diagonal) then the eigenvalues of matrix 'A' are just the entries on the main diagonal of 'A'.

EIGENVECTORS: The Eigenvectors corresponding to an eigenvalue λ of a matrix 'A' are the non-zero vectors that satisfy the equ. $\boxed{(\lambda I - A)\mathbf{x} = \mathbf{0}}$.

THEOREM If k is a positive no., λ is an eigenvalue of a matrix 'A' and $\mathbf{x}$ is a corresp. eigenvector, then λ^k is an eigenvalue of 'A^k' and $\mathbf{x}$ is a corresp. eigenvector.

THEOREM: A square matrix 'A' is Invertible iff $\lambda = 0$ is not an eigenvalue of A.

SIMILAR MATRICES: If 'A' and 'B' are square matrices, then we say that 'B' is similar to 'A' if there is an invertible matrix P such that $B = P^{-1}AP$.

NOTE: The similar matrices 'A' and $P^{-1}AP$ have same determinant, same rank, same nullity, same trace, same characteristic polynomial and same eigenvalues.

DIAGONALIZABLE: A square matrix 'A' is said to be Diagonalizable if it is similar to some diagonal matrix; that is, if there exists an invertible matrix 'P' such that $P^{-1}AP$ is diagonal. In this case, the matrix 'P' is said to diagonalize matrix 'A'.

THEOREM: If 'A' is an $n \times n$ matrix, the following statements are equivalent —

(i) Matrix 'A' is diagonalizable.

(ii) Matrix 'A' has 'n' linearly independent eigenvectors.

THEOREM: If an $n \times n$ matrix 'A' has n distinct eigenvalues (the eigenvectors of 'A' are linearly independent), then matrix 'A' is Diagonalizable.

NOTE. A triangular matrix with distinct entries on main diagonal is Diagonalizable.

NOTE ① If $z = a+ib$ is a complex no., then

(i) $|z| = \sqrt{a^2+b^2}$ is called the modulus (or absolute value) of z.

(ii) $\bar{z} = a-ib$ is called the complex conjugate of z.

Where i, called 'iota' has the property $\boxed{i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}}$.

NOTE ② Every vector in C^n can be split into Real & Imaginary parts as —

$V = (a_1+ib_1,\ a_2+ib_2,\ ---,\ a_n+ib_n) = (a_1, a_2, ---, a_n) + i(b_1, b_2, ---, b_n)$

& $\bar{V} = (a_1-ib_1,\ a_2-ib_2,\ ---,\ a_n-ib_n) = (a_1, a_2, ---, a_n) - i(b_1, b_2, ---, b_n)$

NOTE ③ If 'A' is a complex matrix, then $Re(A)$ and $Im(A)$ are the matrices formed from the real and imaginary parts of entries of 'A' and

'$\bar{A}$' is the matrix formed by taking complex conjugate of each entry in 'A'.

THEOREM: If u & v are vectors in C^n and if k is a scalar, then

(i) $\overline{(\bar{u})} = u$

(ii) $\overline{(ku)} = \bar{k}\bar{u}$

(iii) $\overline{(u+v)} = \bar{u} + \bar{v}$

THEOREM: If 'A' is an $m \times k$ complex matrix and 'B' is a $k \times n$ complex matrix, then

(i) $\overline{(\bar{A})} = A$

(ii) $\overline{(A^T)} = (\bar{A})^T$

(iii) $\overline{(AB)} = \bar{A}\bar{B}$

COMPLEX EUCLIDEAN INNER PRODUCT

If $u = (u_1, u_2, ---, u_n)$ and $v = (v_1, v_2, ---, v_n)$ are vectors in C^n, then

$$\boxed{u.v = u_1\bar{v}_1 + u_2\bar{v}_2 + --- + u_n\bar{v}_n}$$

Euclidean Norm on C^n is defined as $\|u\| = \sqrt{u.u} = \sqrt{u_1\bar{u}_1 + u_2\bar{u}_2 + -- + u_n\bar{u}_n}$

A vector 'u' is called a Unit Vector in C^n if $\|u\| = 1$.

and two vectors u & v are said to be Orthogonal if $u.v = 0$.

THEOREM If 'A' is a 2×2 matrix with real entries then

the Characteristic Equ. of 'A' is $\boxed{\lambda^2 - [\text{trace}(A)]\lambda + \det(A) = 0}$

THEOREM If 'A' is a Real Symmetric matrix, then 'A' has Real eigenvalues.

CHAPTER 6

INNER PRODUCT: An inner product on a real vector space V is a function that associates a real no. $\langle u,v\rangle$ with each pair of vectors in V in such a way that the fallowing axioms are satisfied —

(i) $\langle u,v\rangle = \langle v,u\rangle$

(ii) $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$

(iii) $\langle ku,v\rangle = k\langle u,v\rangle$

(iv) $\langle v,v\rangle \geq 0$ and $\langle v,v\rangle = 0$ iff $v=0$.

Euclidean Inner Product OR Standard Inner Product on R^n

If $u=(u_1,u_2,\dots,u_n)$ and $v=(v_1,v_2,\dots,v_n)$ are two vectors in R^n, then $\boxed{\langle u,v\rangle = u_1v_1+u_2v_2+\dots+u_nv_n}$

NOTE ① If V is a real inner product space, then the Norm (or Length) of a vector v in V is defined as $\boxed{\|v\| = \sqrt{\langle v,v\rangle}}$

and the distance between two vectors u & v is defined as

$$\boxed{d(u,v) = \|u-v\| = \sqrt{\langle u-v, u-v\rangle}}$$

Inner Product on M_{nn}

If U and V are $n\times n$ matrices, then the formula $\boxed{\langle U,V\rangle = \operatorname{trace}(U^TV)}$ defines an Inner Product on the vector space M_{nn}.

For the 2×2 matrices, $U=\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V=\begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$

we have $\langle U,V\rangle = u_1v_1+u_2v_2+u_3v_3+u_4v_4$

Standard Inner Product on P_n

Let $p=a_0+a_1x+\dots+a_nx^n$ and $q=b_0+b_1x+\dots+b_nx^n$ are polynomials in P_n then the standard inner product on P_n is defined as —

$$\langle p,q\rangle = a_0b_0+a_1b_1+\dots+a_nb_n$$

Norm of a polynomial p relative to this inner product is

$$\|p\| = \sqrt{\langle p,p\rangle} = \sqrt{a_0^2+a_1^2+\dots+a_n^2}$$

Least-Square Solu. of Linear Systems – For every linear system $Ax=b$, the associated normal system $A^TAx=A^Tb$ is consistent and all solu. of this system are least squares solutions of $Ax=b$.

ANGLE BETWEEN VECTORS : Let θ be the angle between u & v in a real inner product space, then $\boxed{\cos\theta = \frac{\langle u, v \rangle}{\|u\| \, \|v\|}}$, $0 \le \theta \le \pi$

ORTHOGONALITY :

Two vectors u & v in an inner product space are called Orthogonal if $\langle u, v \rangle = 0$.

Generalized Theorem of Pythogoras :

If u & v are orthogonal vectors in an inner product space, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 .$$

ORTHOGONAL AND ORTHONORMAL SET :- A set of two or more vectors in a real inner product space is said to be Orthogonal if all pairs of distinct vectors in the set are orthogonal.

An orthogonal set in which each vector has norm 1 is said to be Orthonormal.

THEOREM : If $S = \{v_1, v_2, \cdots, v_n\}$ is an Orthogonal set of non-zero vectors in an inner product space, then the set S is linearly independent.

NOTE : Since an Orthonormal set is Orthogonal, it follows that every orthonormal set is linearly independent.

Orthonormal Basis : In an inner product space, a basis consisting of orthonormal vectors is called an Orthonormal basis.

Co-ordinates Relative to Orthonormal Basis

If $S = \{v_1, v_2, \cdots, v_n\}$ is an orthonormal basis for an inner product space V and if u is any vector in V, then

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \cdots\cdots + \langle u, v_n \rangle v_n$$

so the coordinates of u relative to S are $\langle u, v_1 \rangle, \langle u, v_2 \rangle, \cdots, \langle u, v_n \rangle$.

THEOREM Every non-zero finite dimensional inner product space has an Orthonormal Basis.

Gram-Schmidt Process - To convert a basis $\{u_1, u_2, \cdots, u_r\}$ in an orthogonal basis $\{v_1, v_2, \cdots, v_r\}$, perform the following computations —

Step ① Take $v_1 = u_1$

Step ② Take $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Step ③ Take $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

Continueing the above process —

CHAPTER 7

ORTHOGONAL MATRICES — A square matrix 'A' is said to be Orthogonal if its transpose is the same as its inverse, that is, if $A^{-1} = A^T$

or equivalently if $AA^T = A^TA = I$

THEOREM. The following statements are equivalent for an nxn matrix 'A' —

(i) 'A' is orthogonal.

(ii) The row vectors of 'A' form an orthonormal set in R^n with respect to Euclidean inner product.

(iii) The column vectors of 'A' form an orthonormal set in R^n w.r.t. Euclidean inner product.

PROPERTIES OF ORTHOGONAL MATRICES.

1. The inverse of an Orthogonal matrix is Orthogonal.
2. The product of orthogonal matrices is orthogonal.
3. If matrix 'A' is orthogonal, then $\det(A) = 1$ or -1.

ORTHOGONAL DIAGONALIZATION :

If A & B are square matrices, then we say that A & B are Orthogonally Similar if there is an orthogonal matrix P such that $P^TAP = B$.

If the matrix 'A' is orthogonally similar to some diagonal matrix, say $P^TAP = D$, then we say that 'A' is orthogonally diagonalizable and that 'P' orthogonally diagonalizes 'A'.

CONDITIONS FOR ORTHOGONAL DIAGONALIZABILITY

THEOREM : If 'A' is an nxn matrix, then the following statements are equivalent —

(i) 'A' is orthogonally diagonalizable.

(ii) 'A' has an orthonormal set of n eigenvectors.

(iii) 'A' is symmetric.

PROPERTIES OF SYMMETRIC MATRICES.

THEOREM : If 'A' is symmetric matrix, then

(i) The eigenvalues of 'A' are all real numbers.

(ii) Eigenvectors from different eigenspaces are Orthogonal.

QUADRATIC FORMS : If 'A' is symmetric nxn matrix and 'x' is an nx1 column vector of variables, then we call the function $Q_A(x) = x^TAx$, the Quadratic form associated with 'A'.

The general quadratic form on R^2 i.e., $a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2$ can be expressed in matrix form as —

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{i.e., } x^TAx$$

and the general quadratic form on R^3 i.e., $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3$ can be expressed as —

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{i.e., } x^TAx$$

CONJUGATE TRANSPOSE OF A MATRIX : If 'A' a complex matrix, then the Conjugate Transpose of 'A', denoted by A^* is defined as —

$$A^* = (\bar{A})^T = \overline{(A^T)}$$

PROPERTIES

(i) $(A^*)^* = A$

(ii) $(A+B)^* = A^* + B^*$

(iii) $(kA)^* = \bar{k}A^*$

(iv) $(AB)^* = B^*A^*$

Real Matrices	Complex Matrices
Symmetric if $A^T = A$	Hermitian if $A^\theta = A$
Skew-symmetric if $A^T = -A$	Skew-Hermitian if $A^\theta = -A$
Orthogonal if $AA^T = I$	Unitary if $AA^\theta = I$

HERMITIAN MATRIX

A square complex matrix 'A' is said to be Hermitian if $A^* = A$.

PROPERTIES

① The Eigenvalues of a Hermitian matrix are real numbers.

② If 'A' is a Hermitian matrix, then eigenvectors from different eigenspaces are Orthogonal.

UNITARY MATRIX

A square complex matrix 'A' is said to be Unitary if $A^{-1} = A^*$.

or equivalently $AA^* = A^*A = I$.

THEOREM If 'A' is an $n \times n$ matrix with complex entries, then following are Equivalent —

(i) 'A' is Unitary.

(ii) $||Ax|| = ||x||$, $\forall x \in C^n$.

(iii) $Ax \cdot Ay = x \cdot y$, $\forall x, y \in C^n$.

(iv) The column vectors of 'A' form orthonormal set in C^n w.r.t. Complex Euclidean inner product.

(v) The row vectors of 'A' form orthonormal set in C^n w.r.t. Complex Euclidean inner product.

SKEW-SYMMETRIC AND SKEW-HERMITION MATRIX

A square matrix 'A' with real entries is defined to be Skew-symmetric if $A^T = -A$.

A skew-symmetric matrix must have zeros on main diagonal and each entry off the main diagonal must be the negative of its mirror Image about main diagonal.

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

A square complex matrix 'A' is said to be Skew-Hermitian if $A^* = -A$.

A skew-hermitian matrix must have zeros or pure imaginary numbers on main diagonal and each entry off the main diagonal must be the negative of complex conjugate of its mirror image about the main diagonal.

$$A = \begin{bmatrix} i & 1-i & 5 \\ -1-i & 2i & i \\ -5 & i & 0 \end{bmatrix}$$

CHAPTER 8

GENERAL LINEAR TRANSFORMATION — If $T: V \to W$ is a function from a vector space V to a vector space W, then T is called a Linear Transformation from V to W if the following two properties hold for all vectors u & v in V and for all scalars k —

(i) $T(ku) = kT(u)$ [Homogeneity Property]

(ii) $T(u+v) = T(u) + T(v)$ [Additivity Property]

OR

In combination of (i) & (ii), $T(k_1v_1 + k_2v_2) = k_1T(v_1) + k_2T(v_2)$, where $v_1, v_2 \in V$.

THEOREM: If $T: V \to W$ is a Linear Transformation, then

(i) $T(0) = 0$

(ii) $T(u-v) = T(u) - T(v)$, for all u & v in V.

Example ① Matrix Transformations: $T_A: R^n \to R^m$ is a linear transformation.

Example ② Zero Transformation: The mapping $T: V \to W$ such that $T(v) = 0, \forall v \in V$ is a linear transformation.

Example ③ Identity Operator: The mapping $I: V \to V$ defined by $I(v) = v, \forall v \in V$ is a linear transformation.

Example ④ Dilation and Contraction Operators: If V is a vector space and k is scaler, then $T: V \to V$ given by $T(x) = kx$ is a linear operator on V.

Example ⑤ Transformations on Matrix Spaces: Let M_{nn} be the vector space of all $n \times n$ matrices, then —

(i) $T_1(A) = A^T$ is a linear transformation.

(ii) $T_2(A) = \det(A)$ is not linear.

Example ⑥ Translation is not Linear: If x_0 is a fixed non-zero vector in R^2, then the transformation $T(x) = x + x_0$ is not Linear.

THEOREM: Let $T: V \to W$ be a linear transformation, where V is finite-dimensional. If $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for V, then the image of any vector v in V can be expressed as

$$T(v) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n)$$

where $c_1, c_2, \cdots, c_n$ are the coefficients required to express v as a linear combination of vectors in S.

KERNEL AND RANGE OF LINEAR TRANSFORMATIONS

If $T: V \to W$ is a linear transformation, then the set of vectors in V that T maps into '$\mathbf{0}$' is called the Kernel of T and is denoted by $ker(t)$. The set of all vectors in W that are images under T of at least one vector in V is called Range of T and is denoted by $R(t)$.

Example ① Kernel and Range of Zero Transformation: $ker(t) = V$, $R(t) = \{0\}$.

Example ② Kernel and Range of Identity Operator: $ker(I) = \{0\}$, $R(I) = V$.

PROPERTIES OF KERNEL AND RANGE —

THEOREM: If $T: V \to W$ is a linear transformation, then

i) The kernel of T is a subspace of V.

ii) The range of T is a subspace of W.

RANK AND NULLITY OF LINEAR TRANSFORMATIONS

Let $T: V \to W$ be a linear transformation. If the range of T is finite dimensional, then its dimension is called the Rank of T; and if the kernel of T is finite dimensional, then its dimension is called the Nullity of T.

The Rank of T is denoted by $rank(t)$ and the Nullity of T is denoted by $nullity(t)$.

THEOREM: If $T: V \to W$ is a linear transformation from an n-dimensional vector space V to a vector space W, then $rank(t) + nullity(t) = n$.

Note: If $T_A: R^n \to R^m$, then $rank(T_A) + nullity(T_A) = n$.

ONE-TO-ONE LINEAR TRANSFORMATION: If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be One-to-One if T maps distinct vectors in V into distinct vectors in W.

ONTO LINEAR TRANSFORMATION: If $T: V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be Onto if every vector in W is the image of at least one vector in V.

THEOREM If V is a finite-dimensional vector space and if $T: V \to V$ is a linear operator, then the following statements are equivalent —

(i) T is One-to-One.

(ii) $ker(t) = \{0\}$.

(iii) T is Onto [i.e., $R(t) = V$].

Example ① Dilations and Contractions are One-to-One and Onto

Example ② The linear transformations $T_1: P_3 \to R^4$ and $T_2: M_{22} \to R^4$ defined by $T_1(a+bx+cx^2+dx^3) = (a,b,c,d)$ and $T_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a,b,c,d)$ are both one-one & Onto.

Example ③ Differentiation transformation is not-one-to-one since $D(x^2) = D(x^2+1) = 2x$.

<u>Dimension and Linear Transformations</u> — There are two important facts about a linear transformation $T: V \to W$ in the case where V & W are finite-dimensional —

(i) If $\dim(W) < \dim(V)$, then T cannot be One-to-One.

(ii) If $\dim(V) < \dim(W)$, then T cannot be Onto.

<u>ISOMORPHISM</u>: If a linear transformation $T: V \to W$ is both One-to-One & Onto, then T is said to be an Isomorphism and vector spaces V & W are said to be Isomorphic.

<u>THEOREM</u>: Every real n-dimensional vector space is Isomorphic to R^n.

<u>Example ①</u> The linear transformation $T: P_{n-1} \to R^n$ defined by

$$a_0 + a_1 x + \cdots a_{n-1} x^{n-1} \xrightarrow{T} (a_0, a_1, \cdots, a_n)$$

is a Natural Isomorphism from P_{n-1} to R^n.

<u>Example ②</u> The transformation $T: M_{22} \to R^4$ defined by

$$T\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = (a_1, a_2, a_3, a_4)$$

is Natural Isomorphism from M_{22} to R^4.

<u>NOTE</u>: The vector space M_{mn} of $m \times n$ matrices with real entries is Isomorphic to R^{mn}.

<u>COMPOSITIONS OF LINEAR TRANSFORMATIONS</u>

If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations, then the Composition of T_2 with T_1, defind by $T_2 \circ T_1$ is the function from U to W defined by

$$(T_2 \circ T_1)(u) = T_2(T_1(u)) \quad , \text{ where } \mathbf{u} \text{ is a vector in } U.$$

<u>THEOREM</u>: If $T_1: U \to V$ and $T_2: V \to W$ are linear transformations, then

$$(T_2 \circ T_1): U \to W \text{ is also a linear transformation.}$$

<u>INVERSE LINEAR TRANSFORMATIONS</u>

If T is One-to-One, then each vector w in $R(t)$ is the image of a unique vector v in V. This uniqueness allows us to define a new function, called the Inverse of T and denoted by T^{-1}, that maps w back into v.

<u>THEOREM</u>: If $T_1: U \to V$ and $T_2: V \to W$ are one-to-one linear transformations, then — (i) $T_2 \circ T_1$ is one-to-one.

(ii) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

CHAPTER - 9

LU-DECOMPOSITION :- A factorization of a square matrix A as $A = LU$, where L is lower triangular & U is upper triangular matrix, is called an LU-Decomposition of 'A'.

Procedure for Constructing an LU-Decomposition :-

Step① Reduce the matrix 'A' to row echelon form U by Gaussian elimination without row-interchang

Step② In each position along the main diagonal of L, place the reciprocal of multiplier that introduced leading '1' in that position in U.

Step③ In each position below the main diagonal of L, place the negative of multiplier used to introduce the zero in that position in U.

NOTE: LU-Decompositions are not unique.

Solving Linear Systems Using Method of LU-Decomposition :-

Step① Rewrite the system
$$Ax = b \quad \text{—— ①}$$
as
$$LUx = b \quad \text{—— ②}$$

Step② Define a new $n \times 1$ matrix by $Ux = y$ —— ③

Step③ Use ③ to rewrite ② as
$$Ly = b \quad \text{—— ④}$$
and solve the system ④ for y.

Step④ Substitute y in ③ and solve for x.

Dominant Eigenvalues - If the distinct eigenvalues of a matrix 'A' are $\lambda_1, \lambda_2, \cdots, \lambda_k$ and if $|\lambda_1|$ is larger than $|\lambda_2|, |\lambda_3|, \cdots, |\lambda_k|$, then λ_1 is called a Dominant Eigenvalue of 'A'. Any eigenvector corresponding to a dominant eigenvalue is called Dominant Eigenvector of 'A'.

THEOREM Let 'A' be a symmetric $n \times n$ matrix with a positive dominant eigenvalue λ. If x_0 is a unit vector in R^n that is not orthogonal to the eigenspace corresponding to λ, then the normalized power sequence
$$x_0, \; x_1 = \frac{Ax_0}{\|Ax_0\|}, \; x_2 = \frac{Ax_1}{\|Ax_1\|}, \; \cdots\cdots, \; x_k = \frac{Ax_{k-1}}{\|Ax_{k-1}\|}, \; \cdots\cdots$$
converges to a unit dominant eigenvector, and the sequence
$$Ax_1 \cdot x_1 \; [\text{or } (Ax_1)^T x_1], \; Ax_2 \cdot x_2 \; [\text{or } (Ax_2)^T x_2], \; \cdots\cdots, \; Ax_k \cdot x_k, \; \cdots\cdots$$
converges to the dominant eigenvalue λ.

Power Method with Euclidean Scaling -

The above Theorem provides us with an algorithm for approximating the dominant eigenvalue and a corresponding unit eigenvector of a symmetric matrix A, provided the dominant eigenvalue is positive. This algorithm is called the Power method with Euclidean Scaling.

Singular Values: If 'A' is an $m \times n$ matrix and if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A^TA, then the numbers $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \cdots, \sigma_n = \sqrt{\lambda_n}$ are called Singular Values of 'A'.

CHAPTER (10)

GENERAL LINEAR PROGRAMMING PROBLEM IN TWO VARIABLES.

Find the values of x_1 & x_2 that optimize (either maximize or minimize)

$$z = c_1 x_1 + c_2 x_2 \qquad \text{[Linear Objective function]}$$

subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 \ (\leq, \geq \text{ or } =)\ b_1 \\ a_{21}x_1 + a_{22}x_2 \ (\leq, \geq \text{ or } =)\ b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 \ (\leq, \geq \text{ or } =)\ b_m \end{array}\right\} \text{[Linear Constraints]}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0 \qquad \text{[Non-Negativity Constraints]}$$

NOTE ① A pair of values (x_1, x_2) that satisfy all the constraints is called a Feasible Soln. The set of all feasible solutions determines a subset of x_1x_2-plane called the Feasible Region. A feasible solution that optimizes the objective function is called an Optimal Solution.

NOTE ② The feasible region of a linear programming problem has a boundary consisting of a finite no. of straight line segments. If the feasible region can be enclosed in a sufficiently large circle, it is called Bounded; otherwise it is called Unbounded.

If the feasible region is empty (contains no points), then the constraints are Inconsistent and the linear programming problem has no solution.

Those boundary points of a feasible region that are intersections of two of the straight line boundary segments are called Extreme Points (or Corner Points or Vertex Points).

THEOREM (Maximum and Minimum Values)

If the feasible region of a linear programming problem is non-empty and bounded, then the objective function attains both a maximum and a minimum value and these occur at extreme points of the feasible region. If the feasible region is unbounded, then the objective function may or may not attain a maximum or minimum value; however, if it attains a maximum or minimum value, it does so at an extreme point.

QUESTIONS FROM CHAPTERS 5 to 10

Q 1 Find all the eigenvalues and corresponding eigenvectors of matrix, $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$.

Q 2 Find a matrix P that diagonalizes the matrix, $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$.

Q 3 If $u = (1+i, i, 3-i)$ & $v = (1+i, 2, 4i)$, then find $u.v$, $||u||$ and $||v||$.

Q 4 For $u = (1,2,3)$, $v = (4,4,-4)$ and $w = (2,1,7)$, verify the property of inner product,
$$\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle.$$

Q 5 Compute $\langle U, V\rangle$ using the inner product on $M_{2\times 2}$, where
$$U = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \text{ and } V = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}.$$

Q 6 If P_2 have usual inner product on polynomials and
$$p = 1 + x - 2x^2, \quad q = 2 + x^2 \text{ are polynomials.}$$
Then find – (i) $\langle p, q\rangle$ (ii) $||p||$ (iii) $||q||$

Q 7 Let R^3 have the Euclidean inner product. For which value of k are u & v orthogonal? –
(i) $u = (2,1,3)$, $v = (1,7,k)$
(ii) $u = (k,k,1)$, $v = (k,5,6)$

Q 8 Find the least squares solution of the system of linear equ. $AX = B$,
where $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

OR
$$\boxed{\begin{aligned} 2x_1 - x_2 &= 1 \\ x_1 + 2x_2 &= -1 \\ 2x_2 &= -1 \end{aligned}}$$

Q 9 Show that the vectors $w_1 = (0,2,0)$, $w_2 = (3,0,3)$, $w_3 = (-4,0,4)$ form an orthogonal basis for R^3 with Euclidean inner product and use that basis to find an orthonormal basis by normalizing each vector.

Q 10 If $A = \begin{bmatrix} 1+i & 2i & -6 \\ i & 0 & 1-i \\ 2 & 7-6i & 13 \end{bmatrix}$, then find Conjugate transpose of B, where $B = iA$.

Q 11 Show that $A = \begin{bmatrix} 1 & i & 2+3i \\ -i & -3 & 1 \\ 2-3i & 1 & 2 \end{bmatrix}$ is Hermitian.

Q 12 Show that the matrix $A = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{bmatrix}$ is Unitary.

Q(13) Show that the function $T: R^2 \to R^2$ given by $T(x,y) = (-y, x)$ is Linear Transformation.

Q(14) Show that the map $T: R^2 \to R^2$ given by $T(x,y) = (x+y, x-y)$ is Linear.

Q(15) Consider the basis $S = \{V_1, V_2\}$ of R^2, where $V_1 = (1,1)$, $V_2 = (0,1)$. Let $T: R^2 \to R^2$ be the linear transformation for which $T(V_1) = (2,-1)$, $T(V_2) = (3,1)$. Find a formula for $T(x_1, x_2)$.

Q(16) Find a LU-decomposition of the following matrices —

(i) $\begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} -6 & 2 \\ 6 & 0 \end{bmatrix}$

Q(17) Find the Singular values of the following matrices —

(i) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 0 & 3 \end{bmatrix}$

Q(18) Solve the following LPP by graphical method —

(i) $Max.z = x_1 + 2x_2$

subject to $x_1 + x_2 > 2$

$x_2 < 4$

and $x_1, x_2 \geq 0$.

(ii) $Min.z = 2x_1 - x_2$

subject to $2x_1 + 3x_2 = 12$

$2x_1 - 3x_2 \geq 0$

and $x_1 \geq 0, x_2 \geq 0$.

(iii) $Max.z = x_1 + 3x_2$

subject to $2x_1 + 3x_2 \leq 24$

$x_1 - x_2 \leq 7$

$x_2 \leq 6$

and $x_1, x_2 \geq 0$.
